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Painlevé analysis and Bäcklund transformation in the Kuramoto–Sivashinsky equation

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Abstract. We apply the Painlevé test to the Kuramoto-Sivashinsky non-linear partial differential equation, which describes many interesting fluid motions. The equation passes the test in a weak sense. Although the Painlevé analysis does not yield the general solution, we perform the Bäcklund transformation in order to obtain new particular solutions. We find explicitly all the solutions of the set of equations which define the Painlevé-Bäcklund transformation: they reduce to only one type, namely the already known kink-shaped solitary wave.

1. Introduction

The Kuramoto-Sivashinsky (KS) equation is a model partial differential equation (PDE) frequently encountered in the study of continuous media which exhibits a chaotic behaviour. In its conservative form, it is

$$u_t + uu_x + \mu u_{xx} + \nu u_{xxxx} = 0. \tag{1.1}$$

This describes, for instance, the fluctuation of the position of a flame front, or the motion of a fluid going down a vertical wall, or a spatially uniform oscillating chemical reaction in a homogeneous medium. A recent review can be found in a Les Houches lecture [1]. This equation also arises from the minimal ingredients necessary to observe interesting bifurcations in a simplified equation for a complex amplitude in fluid dynamics [2].

Equation (1.1) is invariant under a Galilean transformation

$$(u, x, t) \to (u + c, x - ct, t). \tag{1.2}$$

We make no assumption on the signs of μ and ν, which can obviously be rescaled to ± 1. The purpose of this work is to look for new analytical solutions using the Painlevé-Bäcklund transformation, which will be described below. Particularly, some special solitary waves which have been numerically observed (see figure 2 of [3] and figure 7 of [4]) cannot be represented by the known kink-shaped analytical solution of Kuramoto and Tsuzuki [5]. In § 2, we recall some generalities on the Painlevé analysis of non-linear partial differential equations (NLPDE); this section can be omitted by readers knowing this technique. Its application to the KS equation is described in § 3.

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In § 4, we apply to the KS equation the Bäcklund formalism arising from the condition for truncating the Painlevé expansion. From the four Painlevé-Bäcklund equations, we build an equivalent set of four equations invariant under a group of homographic transformations; this invariance property enables us to find all the solutions of the set of Painlevé-Bäcklund equations. These solutions reduce to only one class, comprised of the kink-shaped stationary waves mentioned above, obtained from the Kuramoto and Tsuzuki solution by making the Galilean transformation (2.2). The Painlevé analysis does not allow us to find new analytical solutions.

The computer algebra program [6] for Painlevé expansion, Bäcklund transformation and invariant equations applied to NLPDE with a polynomial non-linearity has been written in the AMP language [7].

2. Painlevé analysis: generalities

An essential question in the study of NLPDE is the nature of the singularities of the solutions (poles, branch points or essential singularities) and their position (fixed or movable).

For this purpose, the Painlevé analysis, which has been renewed by Ablowitz *et al* [8] for ordinary differential equations (ODE), has been extended to PDE by Weiss *et al* [9]. It consists in looking for the general solution of the PDE in the form (written here in the case of one dependent and two independent variables):

$$u(x, t) = \varphi(x, t)^p \sum_{j=0}^{+\infty} u_j(x, t)\varphi(x, t)^j \tag{2.1}$$

where p is negative, $\varphi(x, t) = 0$ is the equation of a non-characteristic ($\varphi_x\varphi_t \neq 0$) singular manifold, and the functions u_j have to be determined by substitution of expansion (2.1) in the PDE, which becomes:

$$\sum_{j=0}^{+\infty} E_j(u_0, \ldots, u_j, \varphi)\varphi^{j+q} = 0 \tag{2.2}$$

where q is some negative constant. E_j depends on φ only by the derivatives of φ.

The successive practical steps of Painlevé analysis are the following.

(i) Determine the possible leading orders p by balancing two or more terms of the PDE and expressing that they dominate the other terms.

(ii) Solve equation $E_0 = 0$ for non-zero values of u_0; this may lead to several solutions, called branches.

(iii) Find the resonances, i.e. the values of j for which u_j cannot be determined from equation $E_j = 0$. This last equation has usually the form:

$$\forall j > 0 \qquad E_j \equiv (j+1)P(j)\varphi_x^k\varphi_t^{n-k}u_j + Q(u_0, \ldots, u_{j-1}, \varphi) = 0 \tag{2.3}$$

where n is the order of the PDE, $0 \leq k \leq n$, and P a polynom of degree $n-1$. The values of the resonances are the zeros of P.

(iv) Determine if the resonances are 'compatible' or not. At a resonance, after substitution in (2.3) of the previously computed u_l, $l \leq j-1$, the function Q is either zero, in which case u_j can be arbitrarily chosen and the resonance is said to be compatible, or non-zero, and the expansion (2.1) does not exist for arbitrary φ.

The Painlevé property is characterised by the fact that p is a negative integer and all resonances occur at positive integer values of j and are compatible.

3. Painlevé analysis of the Kuramoto–Sivashinsky equation

We find that the dominant terms are uu_x and νu_{xxxx} (let us remark that it will therefore be forbidden to take the limit $\nu \to 0$ in the KS solution obtained by this analysis in order to recover the solution of the Burgers equation). The leading power is $p=-3$ and the first coefficient is

$$u_0 = 120\nu\varphi_x^3. \tag{3.1}$$

In this case

$$P(j) = (j-6)(j^2-13j+60)\nu \tag{3.2}$$

and the resonances are $j=6$ and $j=\frac{13}{2}\pm\mathrm{i}\frac{1}{2}\sqrt{71}$. Since the three resonances are not all located at positive integers, there is no way to obtain the general solution of equation (1.1) in the form (2.1). Nevertheless, we may still obtain a solution depending on two arbitrary functions. In order to examine the nature of the resonance located at $j=6$, we must compute the coefficients u_j up to $j=6$ and we list here the next three coefficients:

$$u_1 = -180\nu\varphi_x\varphi_{xx} \tag{3.3}$$

$$u_2 = 60\nu\varphi_{xxx} + \tfrac{60}{19}\mu\varphi_x \tag{3.4}$$

$$u_3 = -\varphi_t/\varphi_x + 15\nu(-\varphi_{xxxx}/\varphi_x + 2\varphi_{xx}\varphi_{xxx}/\varphi_x^2 - \varphi_{xx}^3/\varphi_x^3) - \tfrac{30}{19}\mu\varphi_{xx}/\varphi_x. \tag{3.5}$$

As already observed by Fournier and Spiegel [10], the resonance $j=6$ is compatible. We can even show that the compatibility condition, which is identically satisfied, has the following nice structure:

$$E_6 \equiv -(\varphi_x^{-1}E_5)_x - \tfrac{1}{2}(\varphi_x^{-1}(\varphi_x^{-1}E_4)_x)_x - \tfrac{1}{6}(\varphi_x^{-1}(\varphi_x^{-1}(\varphi_x^{-1}E_3)_x)_x)_x \tag{3.6}$$

if u_0, u_1 and u_2 have already been replaced by their expression.

From the above analysis, we conclude that the Painlevé expansion yields a solution depending only on two arbitrary functions φ and u_6, and therefore we do not have the general solution. This is an indication, not a proof, of non-integrability (for an analytical proof of non-integrability of the KS equation, see Nicolaenko *et al* [11]; for a 'computer proof', see Thual and Frisch [12]). Thus the KS equation can be said to pass the Painlevé test not in the strict sense (all resonances occur at positive integers and are compatible), but in a weak sense (all resonances which appear at positive integers are compatible).

4. The Painlevé–Bäcklund transformation for the KS equation

The Painlevé expansion (2.1) representing a solution of equation (1.1) can be truncated [13] at $j=3$:

$$u = 60\nu(\log\varphi)_{xxx} + \tfrac{60}{19}\mu(\log\varphi)_x + u_3 \tag{4.1}$$

provided u_3 and φ satisfy the set of five equations

$$E_j(\varphi, u_3) = 0 \qquad j = 3, \ldots, 7. \tag{4.2}$$

Then equation (4.1) defines an auto-Bäcklund transformation, i.e. a transformation between two solutions u_3 and u of the same equation, and we will call equations (4.2) Painlevé-Bäcklund equations.

Instead of E_j, we will prefer to consider $\varphi_x^{j-7}E_j$, which we will again denote E_j, for it is invariant under the group of linear transformations

$$\varphi \rightarrow a\varphi + b. \tag{4.3}$$

Equation $E_3 = 0$ gives the expression of u_3 in terms of the derivatives of φ (see equation (3.5)). Expression E_6 is functionally dependent on E_3, E_4 and E_5 because of the compatibility of the resonance at $j = 6$. Equation $E_7 = 0$ is simply equation (1) taken for u_3. Substituting the expression of u_3 in the other equations, we obtain four equations depending only on the derivatives of φ, and these four equations have a nice property which considerably simplifies the search of their common solutions. Indeed, expression E_4 is invariant under the group of homographic transformations

$$\varphi \rightarrow (a\varphi + b)/(c\varphi + d) \tag{4.4}$$

and this is also the case for expressions $\tilde{E}_j$ defined by

$$\tilde{E}_j = \sum_{k=4}^{j} \left(\frac{\varphi_{xx}}{2\varphi_x}\right)^{j-k} \frac{(7-k)!}{(7-j)!(j-k)!} E_k \qquad j = 4, \ldots, 7. \tag{4.5}$$

As a consequence, expressions $\tilde{E}_j$, which we will simply denote E_j from now on, can be expressed only in terms of two homographic invariants, namely the Schwarzian derivative of φ

$$S = \{\varphi, x\} = \varphi_{xxx}/\varphi_x - \tfrac{3}{2}(\varphi_{xx}/\varphi_x)^2 \tag{4.6}$$

and

$$C = -\varphi_t/\varphi_x \tag{4.7}$$

which has the dimension of a velocity. These two invariants are linked by the cross-derivative condition

$$(\varphi_t)_{xxx} = (\varphi_{xxx})_t \tag{4.8}$$

which is expressed as

$$S_t + C_{xxx} + 2C_xS + CS_x = 0. \tag{4.9}$$

Finally, the set of five equations (4.2) is equivalent to:

$$u_3 = C - 15\nu S_x - 30\nu\left(S + \frac{\mu}{19\nu}\right)\frac{\varphi_{xx}}{\varphi_x} - 15\nu\left(\frac{\varphi_{xx}}{\varphi_x}\right)^3 \tag{4.10}$$

$$E_4 \equiv 120\left(6\nu^2 S_{xx} + 4\nu^2 S^2 + \frac{20\mu\nu}{19} S - 2\nu C_x - \frac{11\mu^2}{19^2}\right) = 0 \tag{4.11}$$

$$E_5 \equiv -60\left(7\nu^2 S_{xx} + 3\nu^2 S^2 + \frac{25\mu\nu}{19} S - 3\nu C_x\right)_x = 0 \tag{4.12}$$

$$E_7 \equiv C_t + CC_x + \frac{177\mu}{2\times 19} C_{xx} + \nu(45SC_{xx} + 12C_{xxxx} + 30C_xS_x) + f(E_4, E_5) = 0 \tag{4.13}$$

where $f(E_4, E_5)$ is an expression vanishing with E_4 and E_5.

In the appendix, we show that the only function φ satisfying simultaneously the three invariant equations, $E_4 = 0$, $E_5 = 0$, $E_7 = 0$, is

$$\varphi(x, t) = \frac{\alpha + \beta\, e^{k(x-ct)}}{\gamma + \delta\, e^{k(x-ct)}} \tag{4.14}$$

where c is arbitrary, α, β, γ, δ are arbitrary constants subject to

$$\alpha\delta - \beta\gamma \neq 0 \tag{4.15}$$

and k can take only four values:

$$k^2 = \frac{-\mu}{19\nu} \qquad \text{or} \qquad \frac{11\mu}{19\nu}. \tag{4.16}$$

Defining $u_{\text{sing}}(\varphi)$ as the singular part in φ of the Painlevé expansion (2.1)

$$u_{\text{sing}}(\varphi) = u_0\varphi^{-3} + u_1\varphi^{-2} + u_2\varphi^{-1} \tag{4.17}$$

$$= 60\nu(\log\varphi)_{xxx} + \tfrac{60}{19}\mu(\log\varphi)_x \tag{4.18}$$

we obtain the general solution of the Painlevé-Bäcklund equations and find

$$u_3 = c - \frac{30\mu k}{19} + u_{\text{sing}}(\gamma + \delta\, \mathrm{e}^{k(x-ct)}) \tag{4.19}$$

$$u = c - \frac{30\mu k}{19} + u_{\text{sing}}(\alpha + \beta\, \mathrm{e}^{k(x-ct)}). \tag{4.20}$$

This solution can be interpreted as a triplet

$$(\varphi(\alpha/\beta, \gamma/\delta), u(\alpha/\beta), u_3(\gamma/\delta)) \tag{4.21}$$

characterised by the two parameters α/β, γ/δ, and we cannot choose u_3 and φ independently in order to perform the Bäcklund iteration

$$u^{(i+1)} = u_{\text{sing}}(\varphi^{(i)}) + u^{(i)}. \tag{4.22}$$

In our case, the iteration is simply defined as

$$\alpha_{i+1}/\beta_{i+1} \text{ is arbitrary} \qquad \gamma_{i+1}/\delta_{i+1} = \alpha_i/\beta_i \tag{4.23}$$

which maps a triplet $(\varphi^{(i)}, u^{(i)}, u_3^{(i)})$ as defined by (4.21) onto a triplet $(\varphi^{(i+1)}, u^{(i+1)}, u_3^{(i+1)})$.

Starting from the constant trivial solution of (1.1) for $u_3^{(0)}$ (i.e. $(\alpha_0, \beta_0, \gamma_0, \delta_0)$ with $\gamma_0\delta_0 = 0$), we will forever remain inside the class of a solution u given by (4.20), which is simply the steady solution of Kuramoto and Tsuzuki after having performed a Galilean transformation of velocity c:

$$u^{(i)} = c + (\tfrac{30}{19}\mu k - 15\nu k^3)\tanh(\tfrac{1}{2}\xi^{(i)}) + 15\nu k^3 \tanh^3(\tfrac{1}{2}\xi^{(i)}) \tag{4.24}$$

with

$$\xi^{(i)} = k(x-ct) + \log(\alpha_i/\beta_i). \tag{4.25}$$

We notice that equation (4.24) represents three distinct types of real solutions, namely:

k real, α/β positive:	kink-shaped regular
k real, α/β negative:	kink-shaped singular
k purely imaginary:	periodic singular.

The solution which comes out of the Painlevé analysis is in agreement with the result [14] that the KS equation has no solution which is a rational fraction of more than one exponential whose argument is linear in x and t.

Another similar auto-Bäcklund transformation is found when studying the equation

$$u_t - u_x^2 - 2uu_{xx} + u_{xxxx} = 0 \tag{4.26}$$

(see formula (8.20) of Weiss [13]). The leading power is -2 and the resonances are: $j=8$ compatible, $j=\frac{1}{2}(7\pm i\sqrt{11})$. The Painlevé analysis gives:

$$u_{\text{sing}}(\varphi)=-\tfrac{15}{2}(\log\varphi)_{xx} \tag{4.27}$$

$$u_2=\tfrac{5}{2}S+\tfrac{15}{8}(\varphi_{xx}/\varphi_x)^2 \tag{4.28}$$

$$E_3\equiv\tfrac{15}{2}(2C-5S_x)=0 \tag{4.29}$$

$$E_4\equiv\tfrac{15}{2}(-S^2-2C_x)=0 \tag{4.30}$$

$$E_5\equiv\tfrac{15}{2}(CS+C_{xx}-3SS_x-S_{xxx})=0 \tag{4.31}$$

$$E_6\equiv\tfrac{5}{2}(-\tfrac{3}{2}S^3+S_t+2S_x^2+SS_{xx}+S_{xxxx})=0 \tag{4.32}$$

and it is remarkable that formula (4.5) also describes the transformation from non-invariant to invariant equations, if one changes 4 and 7 to 3 and 6, respectively. The set of four invariant equations has a unique common solution $C=0$, $S=0$, which also satisfies the compatibility condition. The general solution (φ, u, u_2) is:

$$\varphi=(\alpha+\beta x)/(\gamma+\delta x) \qquad u=\tfrac{15}{2}(x+\alpha/\beta)^{-2} \qquad u_2=\tfrac{15}{2}(x+\gamma/\delta)^{-2} \tag{4.33}$$

where α, β, γ, δ are arbitrary constants, $\alpha\delta-\beta\gamma\neq 0$. The Bäcklund iteration is also given by (4.23) and the generated solutions remain in the one-parameter family

$$\{\tfrac{15}{2}(x+c)^{-2},\ c\in\mathbb{R}\}. \tag{4.34}$$

5. Conclusion

We have proven the consistency of the equations which define the Painlevé–Bäcklund transformation and given their general solution. However, the restriction on the form of φ for the consistency conditions to be satisfied does not allow us to build new analytical solutions.

Appendix. General solution of invariant equations

We show here that the three invariant equations $E_4=0$, $E_5=0$, $E_7=0$ (formulae (4.11)–(4.13)) have the only common solution:

$$\varphi(x,t)=\frac{\alpha+\beta\,\mathrm{e}^{k(x-ct)}}{\gamma+\delta\,\mathrm{e}^{k(x-ct)}} \tag{A1}$$

where α, β, γ, δ are arbitrary constants subject to $\alpha\delta-\beta\gamma\neq 0$, k^2 takes one of the two values $-\mu/(19\nu)$ or $11\mu/(19\nu)$, and c is an arbitrary real constant.

Since the first two equations $E_4=0$ and $E_5=0$ do not explicitly depend on t and allow the elimination of one of the two unknown functions (S, C), the method of solution is the following:

(a) eliminate C between equations $E_4=0$ and $E_5=0$, which leads to a new system of two equations easily solvable;

(b) retain only those solutions which also satisfy the cross-derivative condition (3.9) and equation $E_7 = 0$;

(c) for every solution (S, C), determine the function φ.

We start by integrating equation $E_5 = 0$ with respect to x:

$$F_5 \equiv 7\nu^2 S_{xx} + 3\nu^2 S^2 + \tfrac{25}{19}\mu\nu S - 3\nu C_x - \Lambda = 0 \tag{A2}$$

where Λ is a function of t only.

Then, eliminating C_x between equations $E_4 = 0$ and $F_5 = 0$, we get:

$$4\nu^2 S_{xx} + 6\nu^2 S^2 + \frac{10\mu\nu}{19} S + 2\Lambda - \frac{33\mu^2}{19^2} = 0. \tag{A3}$$

This is a second-order ordinary differential equation for S and has two kinds of solutions, general and singular, as discussed below.

The general solution of (A3) is

$$S = -\frac{5\mu}{6 \times 19\nu} - 4\mathcal{P} \tag{A4}$$

where $\mathcal{P}$ is the Weierstrass elliptic function $\mathcal{P}(x - x_0(t), g_2(t), g_3(t))$, x_0 and g_3 are arbitrary functions and

$$g_2 = -\frac{\Lambda}{4\nu^2} + \frac{223\mu^2}{48 \times 19^2 \nu^2}. \tag{A5}$$

The associated value of C_x is

$$C_x = -40\nu\mathcal{P}^2 - \frac{80\mu}{57}\mathcal{P} + 6\nu g_2 - \frac{112\mu^2}{9 \times 19^2 \nu} \tag{A6}$$

which by integration gives

$$C = -\frac{20\nu}{3}\mathcal{P}' + \frac{80\mu}{57}\zeta - \left(\frac{112\mu^2}{9 \times 19^2\nu} - \frac{8}{3}\nu g_2\right)(x - x_0) + g_4 \tag{A7}$$

in which ζ is another Weierstrass elliptic function $\zeta(x - x_0(t), g_2(t), g_3(t))$, $\mathcal{P}'$ is the derivative of $\mathcal{P}$ with respect to its first argument and g_4 is an arbitrary function of t. When we substitute the solution ((A4), (A7)) in the cross-derivative condition (4.9) and express that its Taylor expansion in $(x - x_0(t))$ must identically vanish, we find for the leading term

$$\tfrac{3520}{3}\nu(x - x_0(t))^{-6} + \ldots \tag{A8}$$

which means that this solution must be rejected.

The singular solution $(S_{xx} = 0)$ of (A3), which cannot be obtained from the general solution with particular values of x_0 and g_3, is given by one of the two roots of

$$S^2 + \frac{5\mu}{3 \times 19\nu} S + \frac{\Lambda}{3\nu^2} - \frac{11\mu^2}{2 \times 19^2\nu^2} = 0 \tag{A9}$$

which implies that S is independent of x.

From the set of equations (4.9), (4.11) and (A9), we derive

$$C=-2\nu x\left(S^2+\frac{5\mu}{19\nu}S-\frac{11\mu^2}{4\times 19^2\nu^2}\right)+K_1 \tag{A10}$$

$$S'(t)=-4\nu S\left(S^2+\frac{5\mu}{19\nu}S-\frac{11\mu^2}{4\times 19^2\nu^2}\right) \tag{A11}$$

$$\Lambda=-3\nu^2\left(S^2+\frac{5\mu}{57\nu}S-\frac{11\mu^2}{2\times 19^2\nu^2}\right) \tag{A12}$$

where K_1 is an arbitrary function of t. Since function C is linear in x and function S independent of x, equation $E_7=0$, which reduces to

$$C_t+CC_x=0 \tag{A13}$$

implies that

$$S^2+\frac{5\mu}{19\nu}S-\frac{11\mu^2}{4\times 19^2\nu^2}=0 \tag{A14}$$

and K_1 is a constant, which we denote c. The two values of S are:

$$S_1=\frac{\mu}{2\times 19\nu}\qquad S_2=\frac{-11\mu}{2\times 19\nu}. \tag{A15}$$

The theory of non-linear differential equations (see, e.g., Hille [15] theorem 10.1.1) says that the general solution of equation $\{\varphi, x\}=r$ is

$$\varphi=\frac{\alpha y_1+\beta y_2}{\gamma y_1+\delta y_2} \tag{A16}$$

where α, β, γ, δ are arbitrary constants such that $\alpha\delta-\beta\gamma\neq 0$, and y_1 and y_2 are two independent solutions of the linear equation:

$$y''+\tfrac{1}{2}ry=0 \tag{A17}$$

Denoting $k^2=-2S$, we obtain for φ the announced expression (A1).

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